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On the Asymptotics of a Solution to an Equation with a Small Parameter at Some of the Highest Derivatives

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Abstract—We study the asymptotic behavior of a solution of the first boundary value problem for a second-order elliptic equation in a nonconvex domain with smooth boundary in the case where a small parameter is a factor at only some of the highest derivatives and the limit equation is an ordinary differential equation. Although the limit equation has the same order as the initial equation, the problem is singulary perturbed. The asymptotic behavior of its solution is studied by the method of matched asymptotic expansions.
Keywords: small parameter, asymptotic expansions.

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We consider the first boundary value problem

$$\mathrm{L}_\varepsilon u = \varepsilon(u_{yy} + u_{zz}) + \mathrm{L}_0 u = f(x, y, z), \quad (x, y, z) \in D, \tag{0.1}$$

$$u(x, y, z) = 0, \quad (x, y, z) \in \Gamma. \tag{0.2}$$

Here, $D \subset \mathbb{R}^3$ is a bounded domain with boundary Γ, and L_0 is the ordinary differential operator

$$\mathrm{L}_0 u = u_{xx} + b_1(x, y, z)u_x + a(x, y, z)u. \tag{0.3}$$

Let the parameter ε be positive, and let all the coefficients of equation (0.1), as well as its right-hand side, be infinitely differentiable.

Assume that the boundary Γ of the domain D is smooth but the domain D is nonconvex.

Assume also that there exists a bounded solution of problem (0.1), (0.2), which will be denoted by $u_\varepsilon(x, y, z)$, which satisfies the estimate

$$|u_\varepsilon(x, y, z)| \le C \max_{(x,y,z)\in\overline{D}} |f(x, y, z)|, \tag{0.4}$$

where the constant C is independent of ε. (This condition holds, for example, if $a(x, x, z) \le \alpha < 0$.)

Problems for elliptic equations with a small parameter at the highest derivatives were studied by different authors [1–3]. These papers were devoted to the case when the order of the limit equation was less than the order of the original equation. This fact, in particular, resulted in the singularity of the problem; i.e., the increasing appearance of singularities in the coefficients of the standard expansion with the growth of the approximation order.

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The peculiarity of the problem under consideration is that the small parameter is a factor at only a part of the highest derivatives. Therefore, the limit equation has the same order; in our case, it is a second-order ordinary differential equation. Nevertheless, the problem is singularly perturbed: the coefficients of the standard (outer) expansion have singularities at points of certain sets in the domain D. In particular, since the domain D is nonconvex, such sets include sets of points belonging to lines $z = C_1$, $y = C_2$ (parallel to the x axis) that touch the boundary Γ of the domain D from within, i.e., points of cylindrical surfaces whose generatrices are parallel to the x axis and directrix is the curve along which the generatrices touch the boundary Γ of the domain D from within.

In this paper, we use the method of matched asymptotic expansions [4,5] to construct and prove the asymptotic expansion of the solution $u_\varepsilon(x, y, z)$ as $\varepsilon \to 0$ in some fixed neighborhood of such cylindrical surface.

For the two-dimensional case, i.e., for the equation $\varepsilon u_{zz} + u_{xx} + b_1(x, z)u_x + a(x, z)u = f(x, z)$, the behavior of solutions of the first boundary value problem was investigated for different domains in [6–8]. In particular, the behavior of the solution in a neighborhood of a straight line that is parallel to the x axis and passes through a point where this line meets the boundary of the domain from within was studied in detail in [7]. We will use the results obtained in [7].

1. OUTER EXPANSION

Assume that, in some fixed neighborhood of the origin $\{(x, y, z) \in D,\ x^2+y^2+z^2 < \delta_0,\ \delta_0 > 0\}$, the domain D coincides with the exterior of the paraboloid $x^2 + y^2 \le z$, which lies in the half-space $z > 0$. Denote by D_δ a neighborhood of the x axis: $D_\delta = (x, y, z) \in D,\ |y| < \delta,\ |z| < \delta^2/2$, where δ is a fixed number such that $\delta < \delta_0$.

The domain D_δ is bounded by the surface of the paraboloid $P\colon z = x^2 + y^2$; by the parts Γ_δ^+ and Γ_δ^- of the boundary Γ that lie in the half-spaces $x > 0$ and $x < 0$, respectively; and by the planes $y = \pm\delta$ and $z = \delta^2/2$. We assume that the equation of the boundary Γ_δ^+ has the form $x = \gamma^+(y, z)$ and the equation of the boundary Γ_δ^- has the form $x = \gamma^-(y, z)$. The asymptotic expansion of the solution $u_\varepsilon(x, y, z)$ as $\varepsilon \to 0$ will be constructed in the domain D_δ.

Define $D_\delta^{(1)} = \{x, z > 0,\ y^2 < z,\ \sqrt{z - y^2} < x < \gamma^+(y, z)\}$, $D_\delta^{(2)} = \{x < 0,\ z > 0,\ y^2 < z,\ \gamma^-(z, y) < x < -\sqrt{z - y^2}\}$, and $D_\delta^{(3)} = D_\delta \setminus (\overline{D_\delta^{(1)}} \cup \overline{D_\delta^{(2)}})$.

The outer expansion

$$U(x, y, z, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k u_k(x, y, z) \tag{1.1}$$

of the solution $u_\varepsilon(x, y, z)$ as $\varepsilon \to 0$ is three different asymptotic series (i.e., $u_k(x, y, z) = u_k^{(j)}(x, y, z)$ for $j = 1, 2, 3$) whose coefficients satisfy in each of the domains $D_\delta^{(j)}$ the same recursive systems of equations

$$\begin{aligned} &\mathrm{L}_0 u_0^{(j)} = f(x, y, z), \quad (x, y, z) \in D_\delta^{(j)}, \\ &\mathrm{L}_0 u_k^{(j)} = -\Delta_{y,z} u_{k-1}^{(j)}, \quad k \ge 1, \quad (x, y, z) \in D_\delta^{(j)}, \end{aligned} \tag{1.2}$$

whereas the boundary conditions for each j are defined on the part of the original boundary Γ that coincides (in the fixed domain D_δ) with the boundary of the domain $D_\delta^{(j)}$:

$$u_k^{(1)}(\gamma^+(y,z),y,z)=0, \quad u_k^{(1)}(\sqrt{z-y^2},y,z)=0;$$
$$u_k^{(2)}(-\sqrt{z-y^2},y,z)=0, \quad u_k^{(2)}(\gamma^-(y,z),y,z)=0; \tag{1.3}$$
$$u_k^{(3)}(\gamma^+(y,z),y,z)=0, \quad u_k^{(3)}(\gamma^-(y,z),y,z)=0.$$

Evidently, the coefficients of the outer expansion $u_k(x,y,z)$ are, by definition, discontinuous on the straight lines that are parallel to the x axis and touch the paraboloid P. It is easy to see that the tangency points belong to the parabola $z=y^2$ (in the plane $x=0$) and, thus, the coefficients of the outer expansion are discontinuous on the cylindrical surface Π whose directrix is the parabola $z=y^2$ and generatrices are parallel to the x axis. Moreover, the functions $u_k^{(1)}(x,y,z)$, $u_k^{(2)}(x,y,z)$ have singularities at points of this cylindrical surface.

The generatrix of the cylindrical surface in the plane $y=y_0$ is tangent to the parabola $x^2=z-y_0^2$. The asymptotics of solutions to recursive ordinary differential equations of the form

$$\mathrm{L}_0 u_k = F_k(u_1,u_2,\ldots,u_{k-1})$$

in a neighborhood of a line tangent to a parabola in the case when the original problem is considered in a two-dimensional domain was studied in detail in [7]. Since the asymptotics of solutions to problems (1.2), (1.3) can be investigated similarly, we restrict ourselves to the formulation of the corresponding theorems.

Theorem 1. *The functions $u_k^{(1)}(x,y,z)$ for $\sqrt{z-y^2}\le x\le\gamma^+(z,y)$ have the following asymptotic expansions as $z-y^2\to 0$:*

$$\begin{aligned} u_0^{(1)}(x,y,z) &= \sum_{j=0}^{\infty}(z-y^2)^{\frac{j}{2}}u_{0j}^+(y),\\ u_k(x,y,z) &= (z-y^2)^{-2k}\sum_{j=1}^{\infty}(z-y^2)^{\frac{j}{2}}u_{kj}^+(y,z), \quad k\ge 1. \end{aligned} \tag{1.4}$$

The functions $u_{kj}^+(y,z)$ are infinitely differentiable, and series (1.4) *admit termwise differentiation of any order.*

Theorem 2. *The functions $u_k^{(2)}(x,y,z)$ for $\gamma^-(z,y)\le x\le -\sqrt{z-y^2}$ have the following asymptotic expansions as $z-y^2\to 0$:*

$$\begin{aligned} u_0^{(2)}(x,y,z) &= \sum_{j=0}^{\infty}(z-y^2)^{\frac{j}{2}}u_{0j}^-(y),\\ u_k(x,y,z) &= (z-y^2)^{-2k}\sum_{j=1}^{\infty}(z-y^2)^{\frac{j}{2}}u_{kj}^-(y,z), \quad k\ge 1. \end{aligned} \tag{1.5}$$

The functions $u_{kj}^-(y,z)$ are infinitely differentiable, and series (1.5) *admit termwise differentiation of any order.*

2. INNER EXPANSIONS

In a neighborhood of the surface Π, we pass to a new inner variable $\zeta = (z - y^2)\varepsilon^{-\frac{1}{2}}$ for each fixed y and construct the inner asymptotic expansion of the solution $u_\varepsilon(x, y, z)$ as $\varepsilon \to 0$ in the form

$$V(x, \zeta, y, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{\frac{k}{4}} v_k(x, \zeta, y). \tag{2.1}$$

Proceeding in the standard way, we pass in original equation (0.1) to the variable ζ; expand the coefficients $b(x, y, z) = b(x, y, y^2 + \sqrt{\varepsilon}\zeta)$ and $a(x, y, z) = a(x, y, y^2 + \sqrt{\varepsilon}\zeta)$ and the right-hand side $f(x, y, z) = f(x, y, y^2 + \sqrt{\varepsilon}\zeta)$ into Taylor series in a neighborhood of the cylindrical surface Π, i.e., for $z = y^2$; and equate the coefficients at equal powers of ε. We obtain the system of recurrence relations

$$\begin{cases} \mathrm{L}_1 v_0 = f(x, y, y^2), \quad \mathrm{L}_1 v_1 = 0, \\ \mathrm{L}_1 v_2 = \dfrac{\zeta}{1!}\dfrac{\partial f}{\partial z}(x, y, y^2) - \Big[4y\dfrac{\partial^2 v_0}{\partial\zeta\partial y} + 2\dfrac{\partial v_0}{\partial\zeta}\Big] - \dfrac{\zeta}{1!}\Big[\dfrac{\partial b}{\partial z}(x, y, y^2)\dfrac{\partial v_0}{\partial x} + \dfrac{\partial a}{\partial z}(x, y, y^2)v_0\Big], \\ \mathrm{L}_1 v_3 = -\Big[4y\dfrac{\partial^2 v_1}{\partial\zeta\partial y} + 2\dfrac{\partial v_1}{\partial\zeta}\Big] - \dfrac{\zeta}{1!}\Big[\dfrac{\partial b}{\partial z}(x, y, y^2)\dfrac{\partial v_1}{\partial x} + \dfrac{\partial a}{\partial z}(x, y, y^2)v_1\Big], \\ \mathrm{L}_1 v_{2j} = \dfrac{\zeta^j}{j!}\dfrac{\partial^j f}{\partial z^j}(x, y, y^2) - \Big[4y\dfrac{\partial^2 v_{2j-2}}{\partial\zeta\partial y} + 2\dfrac{\partial v_{2j-2}}{\partial\zeta}\Big] - \dfrac{\partial^2 v_{2j-4}}{\partial y^2} \\ \qquad - \displaystyle\sum_{i=1}^{j}\dfrac{\zeta^i}{i!}\Big[\dfrac{\partial^i b}{\partial z^i}(x, y, y^2)\dfrac{\partial v_{2j-2i}}{\partial x} + \dfrac{\partial^i a}{\partial z^i}(x, y, y^2)v_{2j-2i}\Big], \\ \mathrm{L}_1 v_{2j+1} = -\Big[4y\dfrac{\partial^2 v_{2j-1}}{\partial\zeta\partial y} + 2\dfrac{\partial v_{2j-1}}{\partial\zeta}\Big] - \dfrac{\partial^2 v_{2j-3}}{\partial y^2} \\ \qquad - \displaystyle\sum_{i=1}^{j}\dfrac{\zeta^i}{i!}\Big[\dfrac{\partial^i b}{\partial z^i}(x, y, y^2)\dfrac{\partial v_{2j+1-2i}}{\partial x} + \dfrac{\partial^i a}{\partial z^i}(x, y, y^2)v_{2j+1-2i}\Big]. \end{cases} \tag{2.2}$$

Here,

$$\mathrm{L}_1 = (1 + y^2)\frac{\partial^2}{\partial\zeta^2} + \frac{\partial^2}{\partial x^2} + b(x, y, y^2)\frac{\partial}{\partial x} + a(x, y, y^2);$$

i.e., for each fixed y, the operator L_1 is elliptic in the variables ζ and x.

The equation of the boundary $z = x^2 + y^2$ of the paraboloid P for fixed y takes the form $\varepsilon^{\frac{1}{2}}\zeta = x^2$, or, which is the same,

$$x = \pm\varepsilon^{\frac{1}{4}}\sqrt{\zeta}, \quad \zeta > 0; \tag{2.3}$$

i.e., for fixed y, the boundary of P turns into parabola (2.3) on the plane (x, ζ), which, as $\varepsilon \to 0$, tends to the cut $x = \pm 0$, $\zeta > 0$ on this plane.

The equations for the boundaries $x = \gamma^{\pm}(z, y)$ take the form

$$x = \gamma^{\pm}(y, z) = \gamma^{\pm}(y, y^2 + \varepsilon^{\frac{1}{2}}\zeta) = \gamma^{\pm}(y, y^2) + \sum_{j=1}^{\infty}\gamma_j^{\pm}(y)\varepsilon^{\frac{i}{2}}\zeta^i,$$

where $\gamma_j^{\pm}(y)$ are infinitely differentiable functions defined by the values of the functions $\gamma^{\pm}(y,z)$ and their derivatives with respect to z on the cylindrical surface Π, i.e., for $z = y^2$.

Thus, for fixed y, as $\varepsilon \to 0$, the domain where equation (2.2) is considered becomes an infinite band with a cut along the positive half-line

$$\Omega(y) = \Big(x, \zeta \colon \gamma^-(y, y^2) < x < \gamma^+(y, y^2),\ \ 0 < \theta < 2\pi\Big), \tag{2.4}$$

where θ is the polar angle on the plane (x, ζ).

In order to write boundary conditions for the coefficients $v_k(x, \zeta, y)$, we require asymptotic series (2.1) to formally satisfy the boundary condition of the original problem on the branches of parabola (2.3), i.e., for $x = \varepsilon^{\frac14}\sqrt{\zeta}$ and $x = -\varepsilon^{\frac14}\sqrt{\zeta}$. We obtain

$$0 = \sum_{k=0}^{\infty} \varepsilon^{\frac{k}{4}} v_k(\pm\varepsilon^{\frac14}\sqrt{\zeta}, \zeta, y) = \sum_{k=0}^{\infty} \varepsilon^{\frac{k}{4}} \Big[\sum_{i=0}^{\infty} \frac{1}{i!}\varepsilon^{\frac{i}{4}} \frac{\partial^i v_k}{\partial x^i}(\pm 0, \zeta, y)\varepsilon^{\frac{i}{4}}(\pm\sqrt{\zeta})^i\Big].$$

Equating to zero the coefficients at powers of ε, we obtain the equations

$$\begin{cases} v_0(+0, \zeta, y) = 0, \quad v_1(+0, \zeta, y) = -\dfrac{\partial v_0}{\partial x}(+0, \zeta, y)\sqrt{\zeta}, \\[2ex] v_2(+0, \zeta, y) = -\dfrac{\partial v_1}{\partial x}(+0, \zeta, y)\sqrt{\zeta} - \dfrac{1}{2!}\dfrac{\partial^2 v_0}{\partial x^2}(+0, \zeta, y)\zeta, \ \ldots, \\[2ex] v_k(+0, \zeta, y) = -\displaystyle\sum_{l=1}^{k} \frac{1}{l!}\frac{\partial^l v_{k-l}}{\partial x^l}(+0, \zeta, y)\zeta^{\frac{l}{2}} \end{cases} \tag{2.5}$$

on the upper side of the cut and

$$\begin{cases} v_0(-0, \zeta, y) = 0, \quad v_1(-0, \zeta, y) = \dfrac{\partial v_0}{\partial x}(-0, \zeta, y)\sqrt{\zeta}, \\[2ex] v_2(-0, \zeta, y) = -\dfrac{\partial v_1}{\partial y}(-0, \zeta, y)(-\sqrt{\zeta}) - \dfrac{1}{2!}\dfrac{\partial^2 v_0}{\partial x^2}(-0, \zeta, y)\zeta, \ \ldots, \\[2ex] v_k(-0, \zeta, y) = -\displaystyle\sum_{l=1}^{k} \frac{1}{l!}\frac{\partial^l v_{k-l}}{\partial x^l}(-0, \zeta, y)(-1)^l\zeta^{\frac{l}{2}} \end{cases} \tag{2.6}$$

on lower side of the cut.

Further, we require asymptotic series (2.1) to formally satisfy the following boundary condition of the original problem on the boundaries $x = \gamma^{\pm}(y, z)$:

$$0 = \sum_{k=0}^{\infty} \varepsilon^{\frac{k}{4}} v_k(\gamma^{\pm}(y, z), \zeta, y) = v_k(\gamma^{\pm}(y, y^2) + \sum_{j=1}^{\infty} \gamma_j^{\pm}(y)\varepsilon^{\frac{i}{2}}\zeta^i)$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{m!}\frac{\partial^m v_k}{\partial x^m}(\gamma^{\pm}(y, y^2)\Big[\sum_{i=0}^{\infty} \varepsilon^{\frac{i}{2}}\zeta^{\frac{i}{2}}\gamma_i^{\pm}\Big]^m(y).$$

We obtain the boundary conditions for the coefficients $v_k(x,\zeta,y)$ on the straight lines $x=\gamma^{\pm}(y,y^2)$

$$\begin{cases} v_0(\gamma^{\pm}(y,y^2),\zeta,y)=0, \quad v_1(\gamma^{\pm}(y,y^2),\zeta,y)=0, \\ v_3(\gamma^{\pm}(y,y^2),\zeta,y)=-\zeta\gamma_1^{\pm}(y)\dfrac{\partial v_1}{\partial x}(\gamma^{\pm}(y),\zeta,y)), \ \ldots, \\ v_k(\gamma^{\pm}(y,y^2),\zeta,y)=S_k^{\pm}(\zeta,y), \end{cases} \tag{2.7}$$

where the boundary function $S_k^{\pm}(\zeta,y)$ has the form

$$S_k^{\pm}(\zeta,y)=\sum_{j,l,s}\beta_{ljs}\frac{\partial^l v_{k-2j}}{\partial x^l}(\gamma^{\pm}(y),\zeta,y)\zeta^s, \quad j\le\left[\frac{k}{2}\right], \quad l+s\le k-2.$$

Thus, in the band $\Omega(y)$ (see (2.4)), it is necessary to construct the functions $v_k(x,\zeta,y)$ that solve for each fixed y elliptic equations (2.2), in which the coefficients and right-hand sides are infinitely differentiable with respect to the parameter y, and satisfy conditions (2.5)–(2.7) on the boundary of this band.

Similarly to the two-dimensional case [7, Sect. 2], since the boundary of $\Omega(y)$ is nonsmooth (it has the cut $x=\pm 0$), the functions $v_k(x,\zeta,y)$, starting with certain k, have singularities at the vertex of the cut $x=\zeta=0$ for every fixed y, i.e., on the parabola $z=y^2,\ x=0$, and the order of the singularities increases with the growth of k. Therefore, the asymptotic expansion becomes ineffective in a neighborhood of the parabola, and it is necessary to construct in this area another asymptotic expansion, which is inner with respect to expansion (2.1). In addition, as in similar problems (see [4, 5, 7]), solutions to problems (2.2), (2.5)–(2.7) are defined nonuniquely in the class of unbounded functions; hence, asymptotic expansion (2.1) can be constructed only after the investigation of the asymptotic behavior of the solution $u_\varepsilon(x,y)$ in a neighborhood of the parabola $z=y^2,\ x=0$.

The equation for outer expansion (1.1) is integrated along straight lines parallel to the x axis and tangent to the surface of the paraboloid P at points of the parabola $z=y^2,\ x=0$. In a neighborhood of this parabola, we pass for each fixed y to new inner variables $\eta=x/\varepsilon^{\frac{1}{2}},\ \xi=(z-y^2)/\varepsilon$ and construct a new inner asymptotic expansion of the solution $u_\varepsilon(x,y)$ as $\varepsilon\to 0$ in the form

$$W(\eta,\xi,y,\varepsilon)=\sum_{k=1}^{\infty}\varepsilon^{\frac{k}{4}}w_k(\eta,\xi,y). \tag{2.8}$$

In order to obtain recurrence relation for finding the coefficients $w_k(\xi,\eta,y)$, we rewrite the coefficients $b(x,y,z)$ and $a(x,y,z)$ and the right-hand side $f(x,y,z)$ of original equation (0.1) in the form $b(\varepsilon^{1/2}\eta,y,y^2+\varepsilon\xi)$, $a(\varepsilon^{1/2}\eta,y,y^2+\varepsilon\xi)$, $f(\varepsilon^{1/2}\eta,y,y^2+\varepsilon\xi)$; expand them into Taylor series in a neighborhood of the point $(0,y,y^2)$; and pass in the obtained expansions to the inner variables η and ξ:

$$\begin{aligned} b(x,y,z)&=\sum_{j=0}^{\infty}\varepsilon^{\frac{j}{2}}g_j^{(1)}(\eta,\xi;y), \\ a(x,y,z)&=\sum_{j=0}^{\infty}\varepsilon^{\frac{j}{2}}g_j^{(2)}(\eta,\xi;y), \\ f(x,y,z)&=\sum_{j=0}^{\infty}\varepsilon^{\frac{j}{2}}g_j^{(3)}(\eta,\xi;y). \end{aligned} \tag{2.9}$$

It is easy to verify that $g_j^{(p)}(\eta,\xi;y)$ is a polynomial of degree j of the form

$$g_j^{(p)}(\eta,\xi;y)=\sum_l \alpha_{jl}^{(p)}(y)\eta^{j-2l}\xi^l, \quad 0\le l\le\left[\frac{j}{2}\right],$$

and its coefficients are infinitely differentiable functions of the parameter y.

In the standard way, we pass in original equation (0.1) to the inner variables η and ξ, replace the coefficients $b(x,y,z)$ and $a(x,y,z)$ and the right-hand side $f(x,y,z)$ by expansions (2.9), and equate the coefficients at equal powers of ε. We find that the coefficients $w_k(\eta,\xi,y)$ of inner expansion (2.8) satisfy for each fixed y the system of recurrence relations

$$\begin{cases} \mathrm{L}_2 w_1=0, \quad \mathrm{L}_2 w_2=0, \\ \mathrm{L}_2 w_3=-b(0,y,y^2)\dfrac{\partial w_1}{\partial\eta}, \quad \mathrm{L}_2 w_4=f(0,y,y^2)-b(0,y,y^2)\dfrac{\partial w_2}{\partial\eta}, \\ \mathrm{L}_2 w_5=-b(0,y,y^2)\dfrac{\partial w_3}{\partial\eta}-\dfrac{\partial b}{\partial x}(0,y,y^2)\eta\dfrac{\partial w_1}{\partial\eta}-a(0,y,y^2)w_1+2\dfrac{\partial w_1}{\partial\xi}, \\ \mathrm{L}_2 w_6=\dfrac{\partial f}{\partial x}(0,y,y^2)\eta-b(0,y,y^2)\dfrac{\partial w_4}{\partial\eta}-\dfrac{\partial b}{\partial x}(0,y,y^2)\eta\dfrac{\partial w_2}{\partial\eta}-a(0,y,y^2)w_2+2\dfrac{\partial w_2}{\partial\xi}, \\ \mathrm{L}_2 w_{2m+1}=-\sum\limits_{j=0}^{m-1} g_j^{(1)}(\xi,\eta,y)\dfrac{\partial w_{2m-1-2j}}{\partial\eta}-\sum\limits_{j=0}^{m-2} g_j^{(2)}(\xi,\eta,y)w_{2m-3-2j} \\ \qquad +2\dfrac{\partial w_{2m-3}}{\partial\xi}+4y\dfrac{\partial^2 w_{2m-5}}{\partial\xi\partial\eta}-\dfrac{\partial^2 w_{2m-7}}{\partial y^2}, \\ \mathrm{L}_2 w_{2m}=g_{m-2}^{(3)}(\eta,\xi;y)-\sum\limits_{j=0}^{m-2} g_j^{(1)}(\xi,\eta,y)\dfrac{\partial w_{2m-2-2j}}{\partial\eta}-\sum\limits_{j=0}^{m-3} g_j^{(2)}(\xi,\eta,y)w_{2m-4-2j} \\ \qquad +2\dfrac{\partial w_{2m-4}}{\partial\xi}+4y\dfrac{\partial^2 w_{2m-6}}{\partial\xi\partial\eta}-\dfrac{\partial^2 w_{2m-8}}{\partial y^2}. \end{cases} \tag{2.10}$$

Here,

$$\mathrm{L}_2=(1+y^2)\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}.$$

Relations (2.10) are elliptic equations in the variables η and ξ, and the variable y is a parameter. For each fixed y, the boundary of the paraboloid becomes the parabola $\xi=\eta^2$; thus, equations (2.10) are considered in the unbounded domain $\Omega_1=\mathbb{R}^2\setminus\big(\eta,\xi\colon \xi>0,\ \eta^2<\xi\big)$, which is the exterior of the domain lying in the half-plane $\xi>0$ and bounded by the parabola $\xi=\eta^2$.

We require asymptotic series (2.8) to satisfy the boundary conditions of the original problem and obtain boundary conditions for the functions $w_k(\eta,\xi,y)$:

$$w_k(\eta,\eta^2,y)=0, \quad k\ge 0. \tag{2.11}$$

Relations (2.10), (2.11), in general, do not uniquely define the functions $w_k(\eta,\xi,y)$, i.e., solutions to the problems in the unbounded domain Ω_1. It is necessary to impose additional conditions on these functions as $\eta^2+\xi^2\to\infty$. These conditions must be obtained from conditions for matching asymptotic expansions (2.1) and (2.8).

The construction and matching of the two inner expansions are carried out simultaneously. For the two-dimensional case, this process is cumbersome; it was considered in detail in [7, Sects. 4–6]. In the problem considered here, the construction of inner expansions is similar to the construction in the two-dimensional case, since the variable y is a parameter and the problems are actually two-dimensional. Therefore, we restrict ourselves to a brief presentation of the result.

Let us first introduce some auxiliary notions. Let s and t be Cartesian coordinates, and let λ and ν be the polar radius and polar angle in the plane (s,t). For integer k, we consider the functions

$$U_k^{(1)}(s,t)=\lambda^{\frac{k}{2}}\sin\frac{k\nu}{2},\quad U_k^{(2)}(s,t)=\lambda^{\frac{k}{2}}\cos\frac{k\nu}{2}. \tag{2.12}$$

The functions $U_k^{(j)}(s,t)$ are conjugate harmonic functions in the plane (s,t) cut along the positive half-line s: $\Delta_{st}U_k^{(j)}=0$.

From explicit formulas (2.12), we easily obtain the relations

$$\frac{\partial U_k^{(2)}}{\partial s}=\frac{\partial U_k^{(1)}}{\partial t}=\frac{k}{2}U_{k-2}^{(2)},\quad -\frac{\partial U_k^{(2)}}{\partial t}=\frac{\partial U_k^{(1)}}{\partial s}=\frac{k}{2}U_{k-2}^{(1)} \tag{2.13}$$

and the relations

$$sU_k^{(1)}=\frac{1}{2}U_{k+2}^{(1)}+\frac{1}{2}\lambda^2U_{k-2}^{(1)},\quad sU_k^{(2)}=\frac{1}{2}U_{k+2}^{(2)}+\frac{1}{2}\lambda^2U_{k-2}^{(2)},$$

$$tU_k^{(1)}=\frac{1}{2}U_{k+2}^{(2)}+\frac{1}{2}\lambda^2U_{k-2}^{(2)},\quad tU_k^{(2)}=\frac{1}{2}U_{k+2}^{(1)}-\frac{1}{2}\lambda^2U_{k-2}^{(1)},$$

which can be written in the general form

$$s^mt^pU_k^{(j)}=\sum_{q=0}^{m+p}\alpha_q^{(m,p)}U_{k+2m+2p-4q}^{(l)}\lambda^{2q}, \tag{2.14}$$

where $l=j$ for even p and $l\neq j$ for odd p. If $U_k^{(j)}$ is a harmonic polynomial, i.e., whenever $k=2n$ for $n\geq 0$, the coefficients $\alpha_q^{(m,p)}$ for which $k+2m+2p-4q<0$ should be assumed to be zero.

Let τ be a parameter. We say that a function $v(s,t;\tau)$ of the form $v(s,t;\tau)=\beta(\tau)\lambda^\alpha\Phi(\nu)$, where $\beta(\tau)$ is a function of the parameter τ, has order α.

The set of linear combinations of functions of the form $v(s,t;\tau)=s^mt^pU_k^{(j)}$, where m and p are nonnegative integers and the coefficients of linear combinations are infinitely differentiable functions of the parameter τ, will be denoted by $\mathcal{W}(s,t;\tau)$. The subset of the set $\mathcal{W}(s,t;\tau)$ with elements of fixed order $q/2$ will be denoted by $\mathcal{W}_q(s,t;\tau)$.

It follows from relation (2.14) that a function $v(s,t;\tau)$ from the set $\mathcal{W}_q(s,t;\tau)$ has the form

$$v(s,t;\tau)=\sum_{m,k,j}\beta_{m,k,j}(\tau)\lambda^{2m}U_k^{(j)}(s,t),$$

where $m\geq 0$, $1\leq j\leq 2$, and $k+4m=q$.

Further, if $v(s,t;\tau)\in\mathcal{W}_q(s,t;\tau)$, we find from relations (2.13) and (2.14) that

$$\frac{\partial v}{\partial s}\in\mathcal{W}_{q-2},\quad \frac{\partial v}{\partial t}\in\mathcal{W}_{q-2},\quad \frac{\partial v}{\partial y}\in\mathcal{W}_{q},\quad s^m t^p v(s,t;\tau)\in\mathcal{W}_{q+2m+2p}.$$

Let us return to the functions $v_j(x,y,\zeta)$ and $w_j(\eta,\xi,y)$.

Theorem 3. *For each fixed y, there exist functions $v_j(x,y,\zeta)$ that solve problems* (2.2), (2.5)–(2.7) *and functions $w_j(\eta,\xi,y)$ that solve problems* (2.10), (2.11) *such that the following asymptotic representations are valid*:

$$v_j(x,y,\zeta)=\sum_{j=1}^{\infty}\omega_j^{(l)}(\zeta,x;y),\quad x^2+\zeta^2\to 0,\tag{2.15}$$

$$w_j(\xi,\eta,y)=\sum_{l=0}^{\infty}\omega_j^{(l)}(\xi,\eta;y),\quad \eta^2+\xi^2\to \infty,\tag{2.16}$$

where $\omega_j^{(l)}(s_1,s_2;y)\in\mathcal{W}_{j-l}\Big(\dfrac{s_1}{\sqrt{1+y^2}},s_2;y\Big)$.

The functions $v_j(x,\zeta,y)$ and $w_j(\xi,\eta,y)$ are defined uniquely. Representation (2.15) *and* (2.16) *are termwise differentiable.*

Proof. The proof of this theorem is similar to the proof of [7, Theorem 6.1].

Thus, we have constructed three asymptotic series: outer expansion (1.1), inner expansion (2.1), and inner expansion (2.8). Each of these series is a formal asymptotic solution of original problem (0.1), (0.2) in the corresponding subdomain of D_δ: outside a neighborhood of the cylinder $z=y^2$; in a neighborhood of the cylinder $z=y^2$ outside a neighborhood of its generatrix, the parabola $z=y^2$, $x=0$; and in a neighborhood of the parabola $z=y^2$, $x=0$. The constructed asymptotic series match: for series (1.1) and (2.1), this follows from relations (1.4) and (1.5); for series (2.1) and (2.8), this follows from relations (2.15) and (2.16). Using partial sums of these series

$$U_n(x,y,z,\varepsilon)=\sum_{k=0}^{n}\varepsilon^k u_k(x,y,z),\quad V_{4n}(x,y,\zeta,\varepsilon)=\sum_{k=0}^{4n}\varepsilon^{\frac{k}{4}}v_k(x,y,\zeta),$$

$$W_{4n}(\eta,\xi,y,\varepsilon)=\sum_{k=0}^{4n}\varepsilon^{\frac{k}{4}}w_k(\eta,\xi,y),$$

where n is a sufficiently large positive integer, we construct in the standard way (see [1, 2]) a composite expansion, which will be a formal asymptotic solution of problem (0.1), (0.2) in the whole domain D_δ. To this end, we consider a function $\chi(t)\in C^\infty(-\infty,+\infty)$ such that $\chi(t)\equiv 1$ for $|t|\le 1$ and $\chi(t)\equiv 0$ for $|t|\ge 2$. Fix ν such that $0<\nu<1/2$, and define the function $S_n(x_1,x_2,z,\varepsilon)$ as follows:

$$\begin{gathered}S^{(n)}(x,y,\varepsilon)=W^{(4n)}(\eta,\xi,y,\varepsilon)\chi((z-y^2)\varepsilon^{-\nu})\chi(-(z-y^2)\varepsilon^{-\nu})\chi(x\varepsilon^{-\frac{\nu}{2}})\chi(-x\varepsilon^{-\frac{\nu}{2}})\\
+V^{(4n)}(x,y,\zeta,y,\varepsilon)\chi((z-y^2)\varepsilon^{-\frac{\nu}{2}})\chi(-(z-y^2)\varepsilon^{-\frac{\nu}{2}})\\
\times\big[1-\chi((z-y^2)\varepsilon^{-\nu})\chi(-(z-y^2)\varepsilon^{-\nu})\chi(x\varepsilon^{-\frac{\nu}{2}})\chi(-x\varepsilon^{-\frac{\nu}{2}})\big]\\
+U^{(n)}(x,y,\varepsilon)\big[1-\chi((z-y^2)\varepsilon^{-\frac{\nu}{2}})\chi(-(z-y^2)\varepsilon^{-\frac{\nu}{2}})\big].\end{gathered}$$

Theorem 4. *For all points of the domain D_δ,*

$$|u_\varepsilon(x,y,z) - S^{(n)}(x,y,\varepsilon)| \le M\varepsilon^{n\lambda},$$

where the constant M is independent of ε and λ depends only on ν.

Proof. The proof of theorems of this kind was described in detail in [5, Ch. 4, Sect. 5]; we omit it here.

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